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Reliability

The reliability $R(t)$ of a unit at time t, is the probability that the unit performs its intended function up to time t under the stated operating conditions.

Let X denote the lifetime of the unit, then the reliability of unit at time t is given by

$$R(t) = P(X > t).$$

Hazard Rate

The hazard rate is the instantaneous rate of failure. We can think of the hazard function as an item's propensity to fail in the next short interval of time, given that the item has survived to time t.

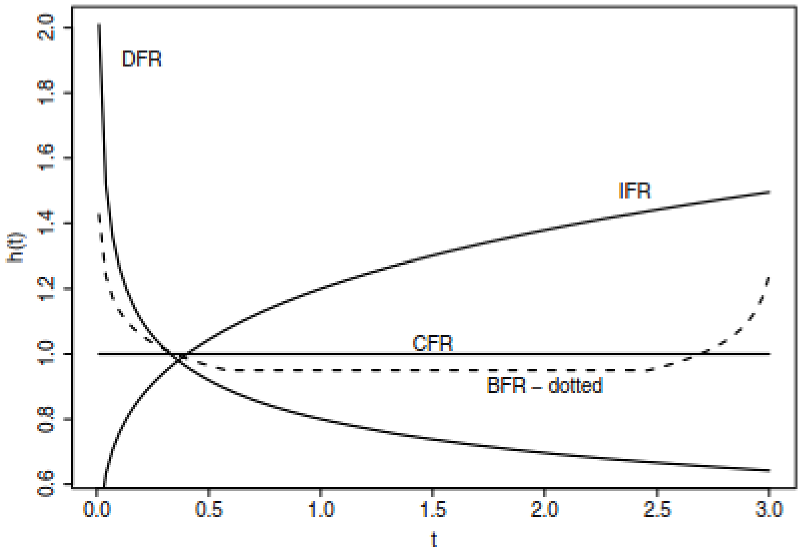
1. Increasing failure rate (IFR): the instantaneous failure rate (hazard) increases as a function of time. We expect to see an increasing number of failures for a given period of time.

2. Decreasing failure rate (DFR): the instantaneous failure rate decreases as a function of time. We expect to see a decreasing number of failures for a given period of time.

3. Bathtub failure rate (BFR): the instantaneous failure rate begins high because of early failures ("infant mortality" or "burn-in" failures), levels off for a period of time ("useful life"), and then increases ("wearout" or "aging" failures).

4. Constant failure rate (CFR): the instantaneous failure rate is constant for the observed lifetime. We expect to see a relatively constant number of failures for a given period of time.

The following figure shows four of the most common types of hazard functions.



Different plots of hazard rates. Dotted lines represents the bath-tub hazard function.

Relation between Hazard rate and Reliability

$$h(t) = \lim_{x \to 0} \frac{F(t+x) - F(t)}{x\bar{F}(t)}$$

Since $\lim_{x \to 0} \frac{F(t+x) - F(t)}{x} = f(t)$

Therefore, we have

$$h(t) = \frac{f(t)}{R(t)} \qquad (1)$$

Integrating, (1) we get

$$\int_0^t h(u)du = \int_0^t \frac{f(u)}{R(u)} du = -\log \bar{F}(t)$$

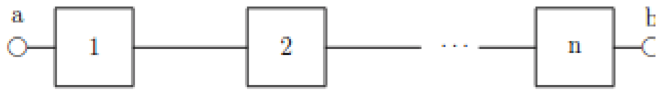
or

$$\bar{F}(t) = \exp\left\{-\int_0^t h(u)du\right\}.$$

System Reliability

In reliability analysis, we often model systems graphically. This provides a visual representation of the components and how they are configured to form a system. One of the most commonly used system representations in risk and reliability analysis is the reliability block diagram .

Series system: A system that functions if and only if all of its components are functioning is series system. Figure below shows the reliability block diagram for a series system.



Suppose that a system with lifetime T consists of n component $C_1, C_2, \ldots, C_n$ are arranged in series network and $T_1, T_2, \ldots, T_n$ denote their respective lifetimes. Then the reliability of ith component at time t is

$$R_i(t) = P(T_i > t)$$

Since components are arranged in series, we have

$$\begin{aligned} R(t) &= P[T > t] \\ &= P[\min(T_1, T_2, \ldots, T_n) > t] \\ &= P[T_1 > t]P[\,T_2 > t] \ldots P[T_n > t] \\ &= \textstyle\prod_{i=1}^{n} P(T_i > t) \\ &= \prod_{i=1}^{n} R_i(t) \end{aligned}$$

That is system reliability equals the product of reliability of all components.

Reliability of Series System in terms of failure rate:

Let $r_i(t)$ denotes the hazard rate of ith component at time t, then

$$R_i(t) = e^{-\int_0^t r_i(u)du}$$

The system reliability is

$$R(t) = \prod_{i=1}^{n} R_i(t)$$

$$e^{-\int_0^t r(u)du} = \prod_{i=1}^{n} e^{-\int_0^t r_i(u)du}$$

That is

$$e^{-\int_0^t r(u)du} = e^{-\int_0^t \sum_{i=1}^{n} r_i(u)du}$$

Thus we get

$$r(t) = \sum_{i=1}^{n} r_i(t)$$

If failure rate is constant, that is $r_i(t) = \lambda_i$

Then $$R_i(t) = e^{\lambda_i(t)}$$

thus

$$R(t) = \prod_{i=1}^{n} e^{\lambda_i(t)} = e^{-\sum_{i=1}^{n} \lambda_i(t)}$$

Thus reliability of system is

$$R(t) = e^{\lambda(t)} \text{ where } \sum_{i=1}^{n} \lambda_i = \lambda.$$

$$MTSF = \int_0^\infty R(t)dt = \int_0^\infty e^{\lambda(t)}dt = \frac{1}{\lambda}\,.$$

Parallel system : A system that functions if at least one of its n components is functioning isa parallel system . Following Figure shows the reliability block diagram for a parallel system.

Suppose that a system with lifetime T consists of n component $C_1, C_2, \dots, C_n$ are arranged in parallel network and $T_1, T_2, \dots, T_n$ denote their respective lifetimes. Then the reliability of ith component at time t is

$$R_i(t) = P(T_i > t)$$

The system reliability is

$$\begin{aligned}
R(t) &= P[T > t] \\
&= 1 - P[T \le t] \\
&= 1 - P[\max(T_1, T_2, \dots, T_n) \le t] \\
&= 1 - P[T_1 \le t]P[\,T_2 \le t] \dots P[T_n \le t] \\
&= 1 - \textstyle\prod_{i=1}^{n} P(T_i \le t) \\
&= 1 - \textstyle\prod_{i=1}^{n} 1 - P(T_i > t)
\end{aligned}$$

$$R(t) = 1 - \prod_{i=1}^{n} (1 - R_i(t)).$$

If failure rate is constant, that is $r_i(t) = \lambda_i$. Then

$$R_i(t) = e^{\lambda_i(t)}$$

Thus, the system reliability is

$$R(t) = 1 - \prod_{i=1}^{n} (1 - R_i(t))$$

$$= 1 - \prod_{i=1}^{n} (1 - e^{\lambda_i(t)})$$

If the components are identical, that is $\lambda_i = \lambda$, then

$$R_i(t) = e^{\lambda(t)}$$

Then

$$R(t) = 1 - (1 - e^{\lambda(t)})^n$$

$$= \binom{n}{1} e^{-\lambda(t)} + \binom{n}{2} e^{-2\lambda(t)} + \cdots$$

$$= \sum_{r=1}^{n} (-1)^{r+1} \binom{n}{r} e^{-r\lambda(t)}.$$

$$MTSF = \int_0^\infty R(t)dt = \int_0^\infty 1 - (1 - e^{-\lambda(t)})^n dt.$$

$$= \frac{1}{\lambda} \int_0^\infty \frac{1 - y^n}{(1 - y)} dy$$

$$= \frac{1}{\lambda} \int_0^\infty (1 + y + y^2 + \ \dots + y^{(n-1)}) dy$$

$$= \frac{1}{\lambda} \left(1 + \frac{1}{2} + \frac{1}{2} + \ \dots + \frac{1}{n}\right)$$

$$= \frac{1}{\lambda} \sum_{i-1}^{n} \frac{1}{i}$$

If the components are identical each having same reliability p(t), then the system reliability

$$R(t) = 1 - (1 - p(t))^n$$

That is

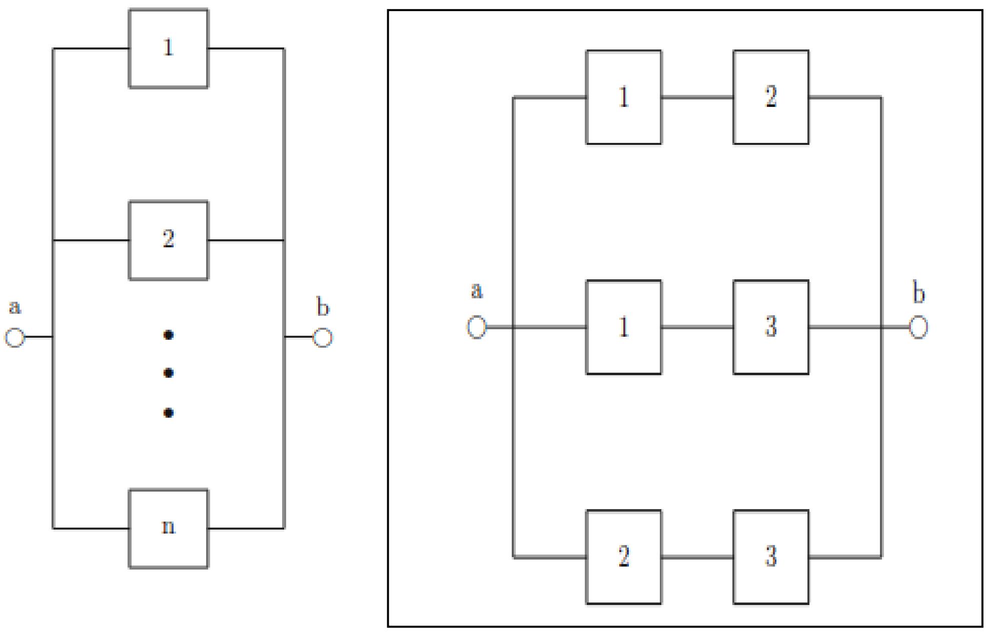
$$nlog\big(1 - p(t)\big) = \log{(1 - R(t))}$$

Thus

$$n = \frac{\log(1 - R(t))}{log(1 - p(t))}.$$

This formula can be used to find the number of component required to achieve the system reliability R(t) when the component reliability is given as p(t).

k-of-n systems: Series and parallel systems are special cases of k-of-n systems. A *k*-of-*n* system functions if at least *k* of its *n* components are functioning. If k = n, we have a series system; if *k* = 1, we have a parallel system. Figure below shows the reliability block diagram for a k-of-n system with k = 2 and n = 3.



Figures: Parallel and *k*-out of –*n* systems.

Suppose that the reliability of each of n components, at time t, is p_t, that is

$$R_i(t) = p_t$$

Then using Binomial law, the probability that out of n components, any k components survive is

$$= \binom{n}{k} (p_t)^k (1 - p_t)^{n-k}$$

Now the probability that at least k survive is

$$R(t) = \sum_{i=k}^{n} \binom{n}{k} (p_t)^i (1-p_t)^{n-i}$$

If failure rate is constant,

Then

$$p(t) = e^{\lambda_i(t)}$$

Structure Function: Structure functions provide another way to summarize the relationships between components in a system. Consider a system with n components. Forthe i^{th} component and time t, define a random variable $x_i = X_i(t)$, so that

$$x_i = \begin{cases} 1 & \textit{if the ith compenent is functioning} \\ 0 & \textit{if the ith compenent is failed}. \end{cases}$$

We can summarize the state of all of the components by a vector $X = (x_1, x_2, \cdots, x_n)$. Some of the 2^n states corresponds to a functioning system ; some corresponds to a failed system. The state of the system is thus a function of X. We call this function the *structure function* and define it as

$$\phi(X) = \begin{cases} 1 \text{ if the system is functioning} \\ 0 \text{ if the system has failed.} \end{cases}$$

Consider a series system, which functions if and only if all of its n components are functioning. Thus $\phi(X) = 1$ if $x_1 = x_2 = \cdots = x_n = 1$, and is 0 otherwise. We can write the following three equivalent expressions:

$$\phi(X) = \begin{cases} 1 \text{ if } \mathrm{x_i} = 1 \text{for all i} \\ 0 \text{ if } \mathrm{x_i} = 0 \text{ for any i,} \end{cases}$$

$$= \min(x_1, x_2, \cdots x_n),$$

$$= \prod_{i=1}^{n} x_i.$$

A parallel system functions is at least one of its components in functioning. Thus, $\phi(X) = 0$ if $x_1 = x_2 = \cdots = x_n = 0$, and is 1 otherwise. We can write the following three equivalent expressions:

$$\phi(X) = \begin{cases} 1 \text{ if } \mathrm{x_i} = 1 \text{ for any i} \\ 0 \text{ if } \mathrm{x_i} = 0 \text{ for all i,} \end{cases}$$

$$= \min(x_1, x_2, \cdots, x_n),$$

$$= 1 - \prod_{i=1}^{n}(1 - x_i).$$

A k-of-n system functions if k or more of its component function. We can write

$$\phi(X) = \begin{cases} 1 \text{ if } \sum_{i=1}^{n} x_i \geq k \\ 0 \text{ if } \sum_{i=1}^{n} x_i < k \end{cases},$$

$$= \sum_j \left(\prod_{i \in A_j} x_i\right)\left[\prod_{i \in A_j^c}(1 - x_i)\right],$$

Where A_j is any subset of $\{1,2,\cdots,n\}$ with at least k elements , and the sum is over all such subsets . For example, the structure function for 2-of-3 system is

$$\phi(X) = \sum_j \left(\prod_{i \in A_j} x_i\right)\left[\prod_{i \in A_j^c}(1 - x_i)\right]$$

$$= x_1x_2(1 - x_3) + x_1x_3(1 - x_2) + x_2x_3(1 - x_1) + x_1x_2x_3$$

$$= x_1x_2 + x_1x_3 + x_2x_3 - 2x_1x_2x_3.$$

Coherent System: A system is coherent if its structure function satisfies the following conditions:

1. $\varphi(0, 0, \ldots, 0) = 0$,
2. $\varphi(1, 1, \ldots, 1) = 1$,
3. $\varphi(x)$ is nondecreasing in each argument.

Minimal Path and Cut Sets

In addition to reliability block diagrams and structure functions, we can use minimal path and cut sets to represent the structure of a system. We call any x for which $\varphi(x) = 1$ a path vector for the system, and any x for which $\varphi(x) = 0$ a cut vector for the structure. The set of component indices corresponding to the functioning (failed) components of a path vector (cut vector) is a path set(cut set).

We denote by $y < x$ if for all i, $y_i \leq x_i$, and for some i, $y_i < x_i$, $i = 1, \ldots n$.

A path vector, x, is a minimal path vector if for every $y < x$, $\varphi(y) = 0$. The minimal path set is the set of components in a minimal path vector that are functioning; that is, a minimal set of components such that if they are all functioning, the system is functioning, but if one of them fails (and all of the components outside the set have failed), then the system fails. A cut vector, x, is a minimal cut vector if for every y > x, φ(y) = 1.

The minimal cut set is the set of components in a minimal cut vector that are failed; that is, a minimal set of components such that if they have all failed, the system has failed, but if one of them is functioning (and all of the components outside the set are functioning), then the system is functioning. We can determine the structure function of a coherent system from either its minimal path sets or its minimal cut sets. Suppose that $\{a_1, a_2, \ldots, a_m\}$ is the collection of all minimal path sets of a coherent system, with x_i being the state variable of the ith component. The system is functioning if all of the components in one or more path sets are functioning. We can think of this

as a parallel arrangement of m sets of components in series. In terms of the minimal path sets, the structure function of the system is

$$\phi(\underline{x}) = 1 - \prod_{i=1}^{m}[1 - \prod_{i \in a_j} x_i)$$

A similar result holds for cut sets. Let $\{b_1, b_2, \ldots, b_k\}$ be the collection of all minimal cut sets of a coherent system, with xi being the state variable of the ith component. The system fails if all of the components in one or more cut sets fail. We can think of this as a series arrangement of k sets of components in parallel. In terms of minimal cut sets, the structure function of the system is

Example Using path sets and cut sets to determine a structure function.

Consider the system in fig. Given below. The minimal path sets are $a_1 = \{1,2\}, a_2 = \{1,3\}$. Using above Eq, The structure function for the system is

$$\phi(X) = 1 - \prod_{j=1}^{2}\left(1 - \prod_{i \in a_j} x_i\right)$$

$$= 1 - (1 - x_1x_2)(1 - x_1x_3)$$

$$= x_1x_2 + x_1x_3 - x_1x_2x_3$$

The minimal cut sets for the system are $b_1 = \{1\}$ and $b_2 = \{2,3\}$. Using Eq.5.4, the structure function for the system is

$$\phi(X) = \prod_{k=1}^{2}(1 - \prod_{i \in b_k}(1 - x_i))$$

$$= (1-(1-x_1))(1-(1-x_1)(1-x_3))$$

$$= x_1(x_2+x_3-x_2x_3)$$

$$= x_1x_2+x_1x_3-x_1x_2x_3.$$

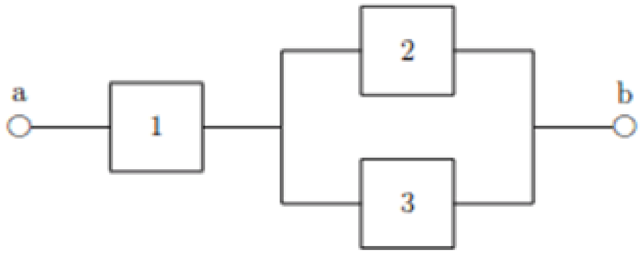


Fig. System with minimal path set $a_1 = \{1,2\}$ and $a_2 = \{1,3\}$.

Relative Importance of Components

For a given coherent system, some components are more important than others in determining whether system functions or not. For example, if a component is in series with rest of the system, then it would seem to be at least as important as any other component in the system.

First suppose we are given the state of each of the remaining components, $(*_i, \underline{x})$. Then we would consider component i more important if

$$\phi(1_i, \underline{x}) - \phi(0_i, \underline{x}) = 1 \qquad (1)$$

rather than

$$\phi(1_i, \underline{x}) = 1 = \phi(0_i, \underline{x}) \text{ or } \phi(1_i, \underline{x}) = 0 = \phi(0_i, \underline{x}).$$

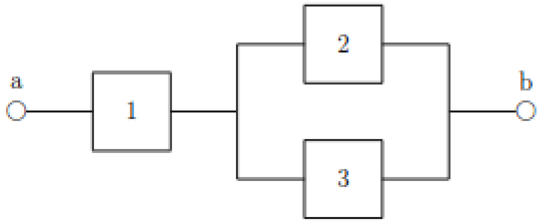
When (1) holds, we call $\phi(1_i, \underline{x})$ a critical path vector for i^{th} component.

Let $\eta_\phi(i) = \sum_{\underline{x|x_i=1}} [\phi(1_i, \underline{x}) - \phi(0_i, \underline{x})]$ denotes total number of critical paths.

Then, the relative importance of i^{th} component is

$$I_\phi(i) = \frac{1}{2^{n-1}} \eta_\phi(i).$$

Example: Determine the importance of various components in following structure.



Here, the structure function of the system is $\phi(\underline{x}) = x_1(x_2 \vee x_3)$.

Since among four outcomes 100, 101, 110 and 111, there are three critical path vectors for component 1, given by (101, 110, and 111). Therefore relative importance of component 1 is

$$I_\phi(1) = \frac{1}{2^2}3 = \frac{3}{4}.$$

However $I_\phi(2) = \frac{1}{2^2}1 = \frac{1}{4}$, since among four outcomes 010, 011, 110, 111 the only one critical path vector for component 2, is 110.

<u>Modular Decomposition</u>

Definition: The coherent system (A, χ) is a module of the coherent system (C, ϕ) if

$$\phi(\underline{x}) = \psi[\chi(\underline{x}^A), \underline{x}^{A^c}],$$

where ψ is a coherent structure function and $A \subseteq C$. The set $A \subseteq C$ is called modular set of (C, ϕ).

Intuitively, a module (A, χ) of (C, ϕ) is a coherent sub-system that acts as if it were just a component. Knowing whether χ is 1 or 0 is as informative as knowing the value of x_i for each i in A, in determining the value of ϕ. In the usual performance diagram of a system, we can identify a module by the fact that it is a cluster of components with one wire leading into it and one wire leading out of it.

Example: Consider a coherent system (C, ϕ) where $\phi(\underline{x}) = x_1(x_2 \vee x_3)(x_4 \vee x_5)$ and $C = \{1, 2, 3, 4, 5\}$. A module of (C, ϕ) is (A, χ) where $A = \{2, 3\}$ and $\chi(\underline{x}^A) = (x_2 \vee x_3)$. We may write

$$\phi(\underline{x}) = \psi[\chi(\underline{x}^A), \underline{x}^{A^c}] = x_1 . \chi .(x_4 \vee x_5).$$

Exponential Distribution

A positive valued random variable X is said to follow exponential distribution if its probability density function is given by

$$f(x;\sigma) = \frac{1}{\sigma}\exp\left\{-\frac{x}{\sigma}\right\}; \quad x > 0, \sigma > 0.$$

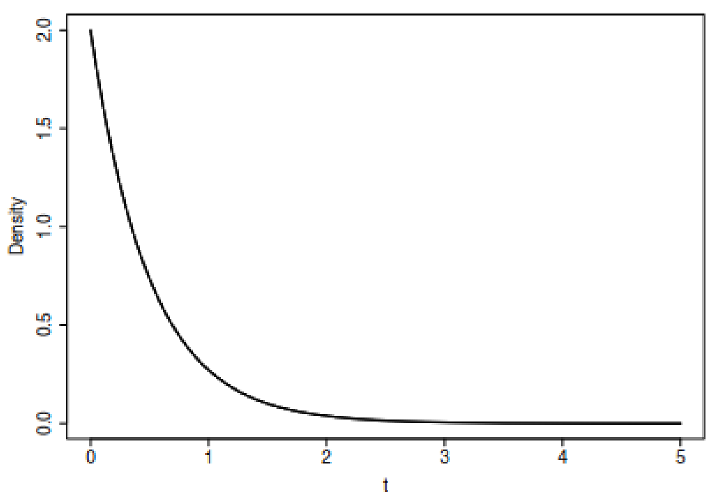
The reliability function, at a mission time t, is

$$R(t) = P(X > t)$$

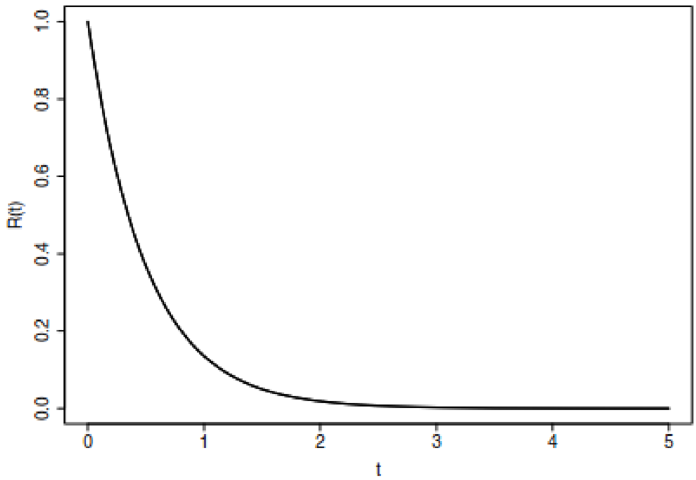
$$= \int_t^{\infty} \frac{1}{\sigma}\exp\left\{-\frac{x}{\sigma}\right\}dx$$

Let $\frac{x}{\sigma} = y \; then \frac{1}{\sigma}dx = dy.$ Thus

$$R(t) = \int_{\frac{t}{\sigma}}^{\infty} e^{-y}dy = e^{-\frac{t}{\sigma}}.$$



The probability density function of exponential random variable with $\sigma = 2$.



The reliability function of exponential random variable with $\sigma = 2$.

The hazard rate is

$$h(t) = \frac{f(t)}{R(t)} = \frac{\frac{1}{\sigma}\exp\left\{-\frac{t}{\sigma}\right\}}{\exp\left\{-\frac{t}{\sigma}\right\}} = \frac{1}{\sigma}.$$

Therefore h(t) is constant $= \frac{1}{\sigma}$.

Mean life=E(X)= $Mean\ life = E(X) = \int_0^\infty x\,\frac{1}{\sigma}\exp\left\{-\frac{x}{\sigma}\right\}dx$

$$= \int_0^\infty y\, e^{-y} dy = \sigma.$$

Maximum Likelihood Estimation

Let $\underline{x} = (x_1, x_2, ..., x_n)$ are iid observations from $\exp(\sigma)$. The likelihood of σ given the observations $\underline{x}$ can be written as follows.

$$L(p, \sigma|\underline{x}) = \prod_{1-1}^{n} f(x_i)$$

$$= \prod_{1-1}^{n}\left\{\frac{1}{\sigma}\exp\left\{-\frac{x}{\sigma}\right\}\right\}$$

$$= \frac{1}{\sigma^n}\exp\left\{-\frac{1}{\sigma}\sum_{1}^{n} x_i\right\}.$$

Taking logarithm, we get

$$\log L(\sigma|\underline{x}) = -n\log\sigma - \frac{1}{\sigma}\sum_{1}^{n} x_i$$

To obtain MLE, we solve the likelihood equation

$$\frac{\partial \log L(\sigma|\underline{x})}{\partial\sigma} = 0$$

That is

$$n/\sigma - \frac{1}{\sigma^2}\sum_{1}^{n} x_i = 0$$

Which gives the MLE of σ

$$\hat{\sigma} = \frac{1}{n}\sum_{1}^{n} x_i$$

Using the invariance property of MLEs, the MLEs of reliability function, hazard rate are

$$\hat{R}(t) = e^{-\frac{t}{\hat{\sigma}}} \quad \text{and} \quad \hat{h}(t) = \frac{1}{\hat{\sigma}}.$$

Weibull Distribution

Definition: A positive valued random variable X is said to follow Weibull distribution if its probability density function is given by

$$f(x; p, \sigma) = \frac{p}{\sigma} x^{p-1} \exp\left\{-\frac{x^p}{\sigma}\right\}; \quad x > 0,\ p, \sigma > 0.$$

where p is shape parameter and σ is scale parameter.

The reliability function is

$$R(t) = P(X > t)$$

$$= \int_t^{\infty} \frac{p}{\sigma} x^{p-1} \exp\left\{-\frac{x^p}{\sigma}\right\} dx$$

Let $\frac{x^p}{\sigma} = y$ *then* $\frac{px^{p-1}}{\sigma} dx = dy.$ Thus

$$R(t) = \int_{\frac{t^p}{\sigma}}^{\infty} e^{-y} dy = e^{-\frac{t^p}{\sigma}}.$$

The hazard rate is

$$h(t) = \frac{f(t)}{R(t)} = \frac{\frac{p}{\sigma} t^{p-1} \exp\left\{-\frac{t^p}{\sigma}\right\}}{\exp\left\{-\frac{t^p}{\sigma}\right\}} = \frac{p}{\sigma} t^{p-1}.$$

Therefore:

When p>1, h(t) is an increasing function of t.

When p=1, h(t) is constant $= \frac{1}{\sigma}$.

When p<1, h(t) is an decreasing function of t.

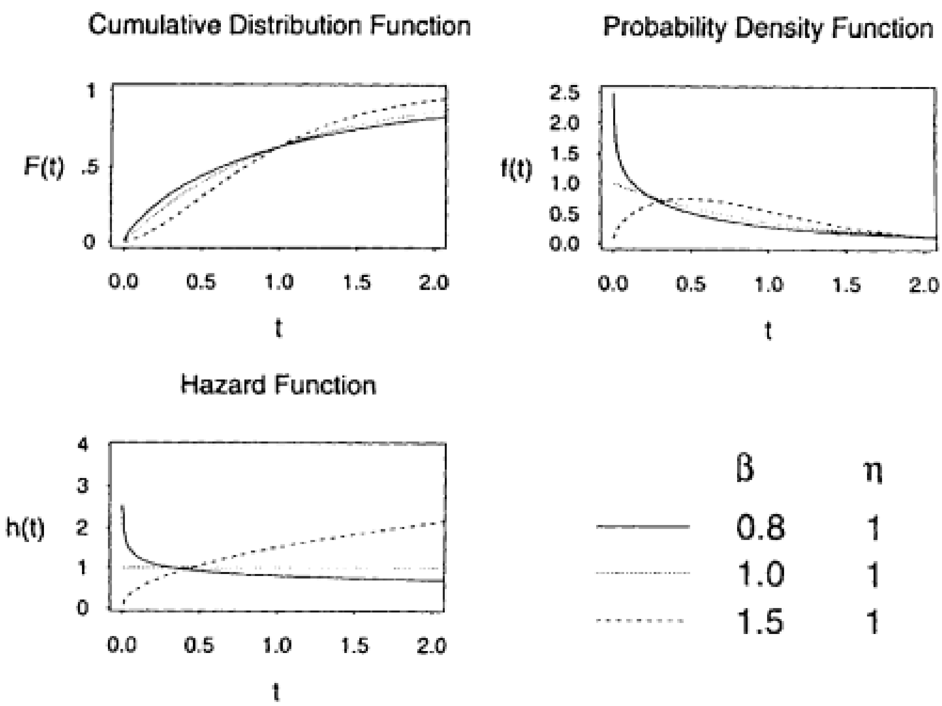


Fig. CDF, pdf and hazard rate functions of Weibull distribution $(p = \beta, \sigma = \eta)$.

$$Mean\ life = E(X) = \int_0^\infty x \frac{p}{\sigma} x^{p-1} \exp\left\{-\frac{x^p}{\sigma}\right\} dx$$

$$= \int_0^\infty \sigma^{1/p} y^{1/p}\, e^{-y} dy$$

$$= \sigma^{1/p} \int_0^\infty y^{\frac{p+1}{p}-1}\, e^{-y} dy$$

$$= \sigma^{1/p} \Gamma\left(\frac{p+1}{p}\right).$$

Maximum Likelihood Estimation

Let $\underline{x} = (x_1, x_2, ..., x_n)$ are iid observations from $w(p, \sigma)$. The likelihood of (p, σ) given the observations $\underline{x}$ can be written as follows.

$$L(p, \sigma|\underline{x}) = \prod_{1-1}^{n} f(x_i)$$

$$= \prod_{1-1}^{n} \left\{ \frac{p}{\sigma} x_i^{\,p-1} \exp\left\{ -\frac{x_i^{\,p}}{\sigma} \right\} \right\}$$

$$= \left(\frac{p}{\sigma} \right)^n \exp\left\{ -\frac{1}{\sigma} \sum_{1}^{n} x_i^{\,p} \right\} \prod_{1-1}^{n} (x_i^{\,p-1})$$

Taking logarithm, we get

$$\log L(p, \sigma|\underline{x}) = n \log p - n \log \sigma - \frac{1}{\sigma} \sum_{1}^{n} x_i^{\,p} + (p-1) \sum_{1}^{n} \log x_i \qquad (1)$$

Case I: When *p* is known

We differentiate (1) with respect to σ the likelihood equation is

$$\frac{\partial \log L(p, \sigma|\underline{x})}{\partial \sigma} = 0$$

That is

$$n / \sigma - \frac{1}{\sigma^2} \sum_{1}^{n} x_i^{\,p} = 0$$

The MLE of σ is

$$\hat{\sigma} = \frac{1}{n} \sum_{1}^{n} x_i^{\,p} \qquad (2)$$

Case II: When both parameters are unknown.

Then we have

$$\frac{\partial \log L(p, \sigma|\underline{x})}{\partial p} = 0$$

That is

$$\frac{n}{p} - \frac{1}{\sigma} \sum_{1}^{n} x_i^{\,p} \log x_i + \sum_{1}^{n} \log x_i = 0$$

We substitute the value of σ from (2) and get

$$\frac{n}{p}+\sum_{1}^{n}\log x_i-\frac{n\sum_{1}^{n}x_i^{\,p}\log x_i}{\sum_{1}^{n}x_i^{\,p}}=0 \tag{3}$$

Equation (3) can be solved through Newton-Raphson method. Using the invariance property of MLEs, the MLE of reliability function, hazard rate and mean life comes out to be

$$\hat{R}(t)=e^{-\frac{t^{\hat{p}}}{\hat{\sigma}}}.$$

The hazard rate is

$$\hat{h}(t)=\frac{\hat{p}}{\hat{\sigma}}t^{\hat{p}-1}.$$

$$MLE\ of\ mean\ life=\hat{\sigma}^{1/p}\Gamma\left(\frac{\hat{p}+1}{\hat{p}}\right).$$

Gamma Distribution

We consider the gamma distribution with pdf given by

$$f(x;p,\sigma)=\frac{1}{\Gamma(p)}\frac{x^{p-1}}{\sigma^p}\exp\left\{-\frac{x}{\sigma}\right\};\quad x>0,\ p,\sigma>0.$$

The reliability function is

$$R(t)=P(X>t)$$

$$=\frac{1}{\Gamma(p)}\int_t^\infty\frac{x^{p-1}}{\sigma^p}\exp\left\{-\frac{x}{\sigma}\right\}dx$$

Let $\frac{x}{\sigma}=y\ then\ dx=\sigma\,dy.$ Thus

$$R(t)=\frac{1}{\Gamma(p)}\int_{\frac{t}{\sigma}}^{\infty}y^{p-1}e^{-y}dy=\frac{1}{\Gamma(p)}\Gamma\left(p,\frac{t}{\sigma}\right)$$

Which is an incomplete gamma function.

The hazard rate is

$$h(t) = \frac{f(t)}{R(t)} = \frac{\frac{1}{\Gamma(p)} \frac{t^{p-1}}{\sigma^p} \exp\left\{-\frac{t}{\sigma}\right\}}{\frac{1}{\Gamma(p)} \Gamma\left(p, \frac{t}{\sigma}\right)}$$

$$= \frac{t^{p-1}}{\sigma^p \; \Gamma\left(p, \frac{t}{\sigma}\right)} \exp\left\{-\frac{t}{\sigma}\right\}$$

Therefore

When p>1, h(t) is an increasing function of t.

When p=1, h(t) is constant $= \frac{1}{\sigma}$.

When p<1, h(t) is an decreasing function of t.

$$Mean\ life = E(X) = \frac{1}{\Gamma(p)} \int_0^\infty x \frac{x^{p-1}}{\sigma^p} \exp\left\{-\frac{x}{\sigma}\right\} dx$$

Let $\frac{x}{\sigma} = y\ then\ dx = \sigma\ dy.$

$$E(X) = \frac{1}{\sigma^p \Gamma(p)} \int_0^\infty (\sigma y)^p \; e^{-y} dy$$

$$= \frac{\sigma}{\Gamma(p)} \int_0^\infty y^{p+1-1} \; e^{-y} dy$$

$$= \frac{\sigma}{\Gamma(p)} \Gamma(p+1).$$

$$= \sigma\ p.$$

Maximum Likelihood Estimation

Suppose that *n* units are put to test and the lifetime of each unit follow gamma distribution with parameter p and σ. Suppose that test is terminated after the failure of all the units. Let the sample $\underline{x} = (x_1, x_2, ..., x_n)$ is observed. The likelihood of (p, σ) given the observations $\underline{x}$ can be written as follows.

$$L(p, \sigma | \underline{x}) = \prod_{1-1}^{n} f(x_i)$$

$$= \prod_{1-1}^{n} \left\{ \frac{1}{\Gamma(p)} \frac{x^{p-1}}{\sigma^p} \exp\left\{ -\frac{x}{\sigma} \right\} \right\}$$

$$= \frac{1}{(\Gamma(p))^n} \frac{1}{\sigma^{np}} \exp\left\{ -\frac{1}{\sigma} \sum_{1}^{n} x_i \right\} \prod_{1-1}^{n} (x_i^{p-1})$$

Taking logarithm, we get

$$\log L(p, \sigma | \underline{x}) = -n \log \Gamma(p) - np \log \sigma - \frac{1}{\sigma} \sum_{1}^{n} x_i + (p-1) \sum_{1}^{n} \log x_i \qquad (1)$$

Case I: When *p* is known

We differentiate (1) with respect to σ the likelihood equation is

$$\frac{\partial \log L(p, \sigma | \underline{x})}{\partial \sigma} = 0$$

That is

$$-\frac{np}{\sigma} + \frac{1}{\sigma^2} \sum_{1}^{n} x_i = 0$$

The MLE of σ is

$$\hat{\sigma} = \frac{1}{np} \sum_{1}^{n} x_i = \frac{\bar{x}}{p} \qquad (2)$$

Case II: When both parameters are unknown.

Then we simultaneously solve

$$\frac{\partial \log L(p, \sigma | \underline{x})}{\partial \sigma} = 0 \text{ and } \qquad \frac{\partial \log L(p, \sigma | \underline{x})}{\partial p} = 0$$

Here

$$\frac{\partial \log L(p, \sigma | \underline{x})}{\partial p} = 0 \text{ implies that}$$

$$-n \frac{d}{dp} \log \Gamma(p) - \sum_{1}^{n} n \log \sigma + \sum_{1}^{n} \log x_i = 0$$

We substitute the value of $\hat{\sigma}$ from (2) and get

$$-n \frac{d}{dp} \log \Gamma(p) - \sum_{1}^{n} n \log\left(\frac{\bar{x}}{p} \right) + \sum_{1}^{n} \log x_i = 0 \qquad (3)$$

Or

$$\frac{d}{dp} \log \Gamma(p) - \log(p) = \frac{1}{n} \sum_{1}^{n} \log x_i - \sum_{1}^{n} \log(\bar{x}) \qquad (4)$$

Now for small values of p can be solved using inverse interpolation and ML estimate of p can be obtained. Further, for large values of p, we can use the following approximation.

$$\frac{d}{dp}\log\Gamma(p) \approx \log(p) - \frac{1}{2p}$$

Then substituting this in (4), we get

$$\log(p) - \frac{1}{2p} - \log(p) = \frac{1}{n}\sum_{1}^{n}\log x_i - \sum_{1}^{n}\log(\bar{x})$$

That is

$$\hat{p} = \frac{1}{2\left\{\sum_{1}^{n}\log(\bar{x}) - \frac{1}{n}\sum_{1}^{n}\log x_i\right\}}$$

after obtaining value of $\hat{p}$, we can get value of $\hat{\sigma}$ from 2.

Log-Normal Distribution

Consider the *lognormal distribution* as a model for failure time data. The lognormal distribution's connection with the normal distribution follows from : if X has a normal distribution, then $T = \exp(X)$ has a lognnormal distribution. Whereas the normal distribution is symmetric about its mean, the log normal distribution is skewed, which makes it potential model for failure times that often exhibit a skewed distribution. The probability density function for a lognormal failure time t is

$$f(t;\mu,\sigma) = \frac{1}{t\sqrt{2\pi\sigma^2}} exp\left(\frac{-1}{2\sigma^2}[\log(t) - \mu]^2\right)$$

(4.12)

Where μ and σ are the mean and statndard deviation of the distribution of the log failure time $x = \log(t)$. We can express the hazard function and reliability function for the lognormal distribution as

$$h(t) = \frac{f(t)}{R(t)} \text{ and}$$

(4.13)

$$R(t) = \int_t^{\bowtie} f(x)dx = 1 - \phi\{[\log(t) - \mu]/\sigma\},$$

Where $f(x)$ is the lognormal probability density function given in Eq. 4.12 and $\phi(\cdot)$ is the standard normal cummulative distribution function. Note that neither the hazard nor reliability functions have closed forms. Historically, the lack of closed form functions is a very measure reason why reliability analysts did not regularly use the lognormal distribution. Today, however software packages routinely evaluate these functions using numerical algorithms. One features of the lognormal distribution is its unique hazard function; the log normal hazard function increase initially and then decreases and approaches zero at very log time. Despite a distribution with decreasing hazard function at long times being untenable; the lognormal distribution has been useful in many applications.

Classes of Distributions

Increasing Failure Rate (IFR)

Suppose $F(x)$ is the distribution of X.

Definition: F is Increasing Failure Rate (IFR) distribution, if for a unit of age t

$$\overline{F}(x|t) = \frac{\overline{F}(t+x)}{\overline{F}(t)} \text{ is decreasing in } t \geq 0.$$

Consequently, we obtain

$$r(t) = \lim_{x\to 0} \frac{1}{x}\left[1 - \frac{\overline{F}(t+x)}{\overline{F}(t)}\right] = \frac{f(t)}{\overline{F}(t)} \text{ is increasing in } t \geq 0.$$

Definition: F is Decreasing Failure Rate (DFR) distribution, if for a unit of age t

$$\overline{F}(x|t) = \frac{\overline{F}(t+x)}{\overline{F}(t)} \text{ is increasing in } t \geq 0.$$

Consequently, we obtain

$$r(t) = \lim_{x\to 0} \frac{1}{x}\left[1 - \frac{\overline{F}(t+x)}{\overline{F}(t)}\right] = \frac{f(t)}{\overline{F}(t)} \text{ is decreasing in } t \geq 0.$$

Definition: F is Increasing Failure Rate Average (IFRA) distribution, if for a unit of age t

$$-\frac{1}{t}\log \overline{F}(t) \text{ is increasing in } t \geq 0.$$

Or

$$\overline{F}(\alpha t) \geq \left(\overline{F}(t)\right)^{\alpha} \qquad 0 < \alpha < 1.$$

Definition: F is Decreasing Failure Rate Average (DFRA) distribution, if for a unit of age t

$$-\frac{1}{t}\log \overline{F}(t) \text{ is decreasing in } t \geq 0.$$

Or

$$\overline{F}(\alpha t) \leq \left(\overline{F}(t)\right)^{\alpha} \qquad 0 < \alpha < 1.$$

Lemma: Let $h(\underline{p})$ be the reliability function of a monotonic system, then

$$h(\underline{p}^{\alpha}) \geq h^{\alpha}(\underline{p}) \qquad 0 < \alpha < 1. \qquad (1)$$

where $\underline{p}^{\alpha} = (p_1^{\alpha}, p_2^{\alpha}, ..., p_n^{\alpha}) \qquad 0 < \alpha < 1.$

Proof: We prove the theorem by mathematical induction.

When n=1, $h(\underline{p}) \geq p$, then

We have $p^{\alpha} \geq \{p\}^{\alpha}$,

and (1) holds.

Now, we assume that (1) holds for all monotonic systems of *(n*-1) components. Then for the monotonic systems of *n* component, we have by pivotal decomposition formula

$$h(\underline{p}^{\alpha}) = p_n^{\alpha}\, h(1_n, \underline{p}^{\alpha}) + (1 - p_n^{\alpha})\, h(0_n, \underline{p}^{\alpha})$$

Since $h(1_n, \underline{p}^{\alpha}) \geq h^{\alpha}(1_n, \underline{p})$ [Each component is IFRA]

and

$$h(0_n, \underline{p}^{\alpha}) \geq h^{\alpha}(0_n, \underline{p})$$

Hence

$$h(\underline{p}^{\alpha}) = p_n^{\alpha}\, h^{\alpha}(1_n, \underline{p}) + (1 - p_n^{\alpha})\, h^{\alpha}(0_n, \underline{p})$$

Now for $0 \leq \alpha \leq 1, 0 \leq \lambda \leq 1 \; and \; 0 \leq x \leq y$, we have

$$x^{\alpha}\ y^{\alpha} + (1 - \lambda^{\alpha}) x^{\alpha} \geq [\lambda y + (1 - \lambda) x]^{\alpha}. \qquad (2)$$

Taking $\lambda = p_n, x = h(0_n, \underline{p}),\ y = h(1_n, \underline{p})$, we obtain using (2) that

$$\begin{aligned} h(\underline{p}^{\alpha}) &\geq p_n^{\alpha}\, h^{\alpha}(1_n, \underline{p}) + (1 - p_n^{\alpha})\, h^{\alpha}(0_n, \underline{p}) \\ &\geq [p_n\, h(1_n, \underline{p}) + (1 - p_n)\, h(0_n, \underline{p})]^{\alpha} \\ &\geq [h(\underline{p})]^{\alpha}. \end{aligned}$$

Now we prove **IFRA closure** theorem.

Theorem: Suppose each independent component of a coherent system has an IFRA distribution, then the system itself has an IFRA distribution.

Proof: Let *F* denotes the system life distribution, while F_i denote the lifetime distribution of i^{th} component.

For $0 < \alpha < 1$

$$\overline{F}(\alpha t) = h[\overline{F}_1(\alpha t), \overline{F}_2(\alpha t), ..., \overline{F}_n(\alpha t)]$$

Since F_i is IFRA then $\overline{F}_i(\alpha t) \geq \overline{F}_i^{\,\alpha}(t)$. Also *h* is increasing in each argument, thus

$$\overline{F}(\alpha t) = h[\overline{F}_1^{\alpha}(t), \overline{F}_2^{\alpha}(t), ..., \overline{F}_n^{\alpha}(t)] \qquad \text{(i)}$$

But we know that

$$h[\overline{F}_1^{\alpha}(t), \overline{F}_2^{\alpha}(t), ..., \overline{F}_n^{\alpha}(t)] \geq h^{\alpha}[\overline{F}_1(t), \overline{F}_2(t), ..., \overline{F}_n(t)] \qquad \text{(ii)}$$

From (i) and (ii), we have

$$\overline{F}(\alpha t)] \geq h^{\alpha}[\overline{F}_1(t), \overline{F}_2(t), ..., \overline{F}_n(t)]$$

That is

$$\overline{F}(\alpha t) \geq \overline{F}^{\alpha}(t).$$

Thus F is IFRA.

Classes of distributions applicable in replacement policies

(NBU) New Better than Used class of distributions

A distribution F is NBU if

$$\overline{F}(x+y) \leq \overline{F}(x)\overline{F}(y)$$

Or

$$\frac{\overline{F}(x+y)}{\overline{F}(x)} \leq \overline{F}(y)$$

That is the conditional survival probability $\overline{F}(x+y)/\overline{F}(x)$ of a unit at age x is less than the corresponding survival probability $\overline{F}(y)$ of a new unit.

(NWU) New Worse than Used class of distributions

A distribution F is NWU if

$$\overline{F}(x+y) \geq \overline{F}(x)\overline{F}(y)$$

Or

$$\frac{\overline{F}(x+y)}{\overline{F}(x)} \geq \overline{F}(y)$$

That is the conditional survival probability $\overline{F}(x+y)/\overline{F}(x)$ of a unit at age x is more than the corresponding survival probability $\overline{F}(y)$ of a new unit.

NBUE (New Better than Used in Expectation) class of distributions

A distribution F is NBUE if

(i) F has finite mean.

(ii) $\int_t^\infty \overline{F}(x)dx \le \mu\, \overline{F}(t)$ $\quad\quad$ *for* $t > 0$.

Where μ is the mean life of a new unit. We note that $\frac{1}{\overline{F}(t)}\int_t^\infty \overline{F}(x)dx$ represent the conditional mean remaining life of a unit of age t. Thus (ii) indicates that used unit of age t has smaller mean remaining life than a new unit if F is NBUE.

NWUE (New Worse than Used in Expectation) class of distributions

A distribution F is NWUE if

(a) F has finite mean.

(b) $\int_t^\infty \overline{F}(x)dx \ge \mu\, \overline{F}(t)$ $\quad\quad$ *for* $t > 0$.

The second condition Indicates that used unit of age t has more mean remaining life than a new unit if F is NBUE.

Hazard Transform: For a coherent system of n components, Let F_i be the life distribution of the i^{th} (i=1, 2, ...,n)component. Then the hazard function of component i is $R_i(t) = -\log \overline{F}_i(t)$. Let R be the system hazard. Then the function η, expressing system hazard in terms of component hazard $\underline{R}(= R_1, R_2, ..., R_n)$ is called the hazard transform $R = \eta(\underline{R})$.

The hazard transform is increasing in each argument. It is continuous and finite and it is super additive, that is,

$$\eta(\underline{R}^{(1)} + \underline{R}^{(2)}) \ge \eta(\underline{R}^{(1)}) + \eta(\underline{R}^{(2)}).$$

Theorem: If each component of a coherent system has an NBU distribution then the system itself has an NBU life distribution.

Proof: Let R_i be the ith component hazard and R the system hazard. If F_i is NBU then by definition

$$\overline{F}_i(s+t) \le \overline{F}_i(s)\overline{F}_i(t)$$

That is

$$\exp\left(-R_i(s+t)\right) \le \exp\left(-R_i(s)\right)\exp\left(-R_i(t)\right)$$

Implies that

$$R_i(s+t) \ge R_i(s)R_i(t)$$

Since η is increasing then

$$\eta\left[\underline{R}(s+t)\right] \ge \eta\left[R(s)+R(t)\right] \qquad (1)$$

Also η is superadditive, then

$$\eta\left[R(s)+R(t)\right] \ge \eta\left[R(s)\right]+\eta\left[R(t)\right] \qquad (2)$$

From (1) and (2)

$$R(s+t) \equiv \eta\left[\underline{R}(s+t)\right] \ge \eta\left[R(s)\right]+\eta\left[R(t)\right] \equiv R(s)+R(t)$$

That is

$$R(s+t) \ge R(s)+R(t).$$

or $\quad \overline{F}(s+t) \le \overline{F}(s)\overline{F}(t)$

That is system lifetime distribution is NBU.

Mixture of Distributions

Suppose that an important quality characteristic of a product being manufactured depends on the amount α of impurity present in the raw material, specifically, the probability distribution of the quality characteristic is F_α. Suppose α itself is random with distribution $G(\alpha)$. Then the resultant distribution F of the quality characteristic is given by

$$F(x) = \int_{-\infty}^{\infty} F_\alpha(x)dG(\alpha).$$

The hazard transform of the mixture $F(x) = \int_{-\infty}^{\infty} F_\alpha(x)dG(\alpha)$ is

$$\eta(\underline{u}) = -\log \int_{-\infty}^{\infty} e^{-u_\alpha} dG(\alpha),$$

where the vector $\underline{u}$ has elements $\underline{u}, 0 \le u_\alpha \le \infty, -\infty \le \alpha \le \infty.$

Moreover

$$R(t)=\eta(\underline{R}(t))=-\log\int_{-\infty}^{\infty}e^{-R_\alpha(t)}dG(\alpha); \qquad 0\le t\le\infty.$$

Where $R_\alpha(t)$ is the hazard function of F_α.

Theorem: Let F(t) be the mixture given by $F(x)=\int_{-\infty}^{\infty}F_\alpha(x)dG(\alpha)$, then

(a)If each F_α is DFR, then F is DFR.

(b) If each F_α is DFRA then F is DFRA.

Proof: (a) Let R_α be the hazard function of F_α. Since each F_α is DFR, it implies that each R_α is concave. So we may prove that is each R_α is concave then R, the hazard function of F will also be concave.

In this direction, we first prove that the hazard transform of the mixture is concave.

By Holder's inequality, we have for $0\le u_\alpha\le\infty, 0\le v_\alpha\le\infty, -\infty\le\alpha\le\infty$ and $0\le a\le 1$,

$$\int e^{-au_\alpha}e^{-(1-a)v_\alpha}dG(\alpha)\le\left\{\int e^{-au_\alpha}dG(\alpha)\right\}^a\left\{\int e^{-v_\alpha}dG(\alpha)\right\}^{(1-a)}$$

Or

$$-\log\int e^{-[au_\alpha+(1-a)v_\alpha]}dG(\alpha)\ge -a\log\int e^{-au_\alpha}dG(\alpha)-(1-a)\log\int e^{-v_\alpha}dG(\alpha)$$

That is,

$$\eta[a\underline{u}+(1-a)\underline{v}]\ge a\eta(\underline{u})+(1-a)\eta(\underline{v}) \qquad (1)$$

Which implies that η is concave.

Since R_α is concave then

$$R_\alpha[at+(1-a)t']\ge aR_\alpha(t)+(1-a)R_\alpha(t')$$

And $\underline{R}$ has elements R_α, then

$$\begin{aligned}R[at+(1-a)t'] &\equiv \eta[\underline{R}[at+(1-a)t']]\\ &\ge \eta[a\underline{R}(t)+(1-a)\underline{R}(t')\\ &\ge a\eta\big(\underline{R}(t)\big)+(1-a)\eta\big(\underline{R}(t')\big)\\ &\ge aR(t)+(1-a)\underline{R}(t')\end{aligned}$$

Implies that F is DFR.

Proof (b): If each F_α is DFRA, we have for $0\le a\le 1$,

$$\overline{F}(\alpha t) \leq \overline{F}^{\alpha}(t).$$

$$\exp(-R_{\alpha}(\alpha t)) \leq \exp(-aR_{\alpha}(t))$$

$$R_{\alpha}(\alpha t) \geq aR_{\alpha}(t)$$

Thus, we can say that we are given (2) for each R_{α} and we have to prove it for each R.

Consider

$$R(\alpha t) \equiv \eta[\underline{R}(at)] \geq \eta[a\underline{R}(t)]$$

Choosing $\underline{v} = 0$ in (1), we get

$$\eta[\underline{aR}(at)] \geq a\eta[\underline{R}(t)] \geq aR(t)$$

Therefore

$$R(\alpha t) \geq aR(t)$$

That is, F is DFRA.

In order to study the various properties of failure rate r(t), it is useful to consider the polya frequency of order 2 (PF_2), which are of considerable independent interest (Karlin, 1968).

Definition : A function h(x), $-\infty < x < \infty$, is PF_2 if

a) $h(x) \geq 0$ for $-\infty < x < \infty$ and
b) for all $-\infty < x_1, x_2 < \infty$ and $-\infty < y_1, y_2 < \infty$,

$$\begin{vmatrix} h(x_1 - y_1) & h(x_1 - y_2) \\ h(x_2 - y_1) & h(x_2 - y_2) \end{vmatrix} \geq 0$$

OR

b') log h(x) is concave on $(-\infty,\infty)$

OR

b'') for fix $\Delta > 0$, $h(x + \Delta)/h(x)$ is decreasing in x for $a < x < b$ where

$a = \inf_{h(y)>0} y$, $b = \sup_{h(y)>0} y$.

Since F IFR is equivalent to $\overline{F}(t+\Delta)/\overline{F}(t)$ decreasing in $-\infty < t < \infty$ for each $\Delta \geq 0$. Thus by b'' F IFR is equivalent to $\bar{F}$ PF$_2$.

Convolution : Let F_1 and F_2 be distribution (not necessarily confined to positive half axis) of X_1 and X_2 respectively. If X_1 and X_2 are independent then the distribution F of $X_1 + X_2$ is given by

$$F(t) = \int_{-\infty}^{\infty} F_1(t-x)dF_2(x)$$

We say that F is the convolution of F_1 and F_2 .

Addition of Life Length :

When a failed component is replaced by a spare, the total life accumulated is obtained by the addition of two life length. Suppose the life of the component 1 is denoted by X_1 having distribution F_1 and life of component 2 is denoted by X_2 with distribution F_2 . Moreover if component 1 fails at any time x preceding t, while component 2 fails during the interval of time (t-x) remaining then F, the distribution of $X_1 + X_2$, is given by

$$F(t) = \int_{0}^{t} F_2(t-x)dF_1(x)$$

Theorem : if F_1 and F_2 are IFR , then their convolution $F(t) = \int_{0}^{t} F_1(t-x)dF_2(x)$ is IFR.

Proof : We know that IFR is equivalent to $\bar{F}$ is PF$_2$. To prove this , we write for $t_1 < t_2$ and $\mu_1 < \mu_2$.

$$D = \begin{vmatrix} \bar{F}(t_1-\mu_1) & \bar{F}(t_1-\mu_2) \\ \bar{F}(t_2-\mu_1) & \bar{F}(t_2-\mu_2) \end{vmatrix}$$

If F_1 has density f_1 and F_2 has density f_2 then

$$\overline{F}(t_1-\mu_1)=\int_0^{t_1-\mu_1}\overline{F}_1(t_1-\mu_1-x)f_2(x)dx=\int_0^{t_1-\mu_1}\overline{F}(t_1-s)f_2(s-\mu_1)ds$$

$$\Theta\ \ \mu_1+x_1=s_1\Rightarrow x_1=s_1-\mu_1\Rightarrow dx_1=ds_1$$

$$D^*=\begin{vmatrix}\int\overline{F}_1(t_1-s)f_2(s-\mu_1)ds & \int\overline{F}_1(t_1-s)f_2(s-\mu_2)ds\\ \int\overline{F}_1(t_2-s)f_2(s-\mu_1)ds & \int\overline{F}_1(t_2-s)f_2(s-\mu_2)ds\end{vmatrix}$$

We use a result by Karlin (1968), known as basic composition formula. According to this formulae,

If $w(x,z)=\int\mu(x,y)u(y,z)dy$ converges absolutely, where( ), Then

$$\begin{vmatrix}w(x_1,z_1) & \Lambda & w(x_1,z_n)\\ \mathrm{M} & \mathrm{O} & \mathrm{M}\\ w(x_n,z_1) & \Lambda & w(x_n,z_n)\end{vmatrix}=\iiint_{y_1<\Lambda<y_n}\begin{vmatrix}\mu(x_1,y_1) & \Lambda & \mu(x_1,y_n)\\ \mathrm{M} & \mathrm{O} & \mathrm{M}\\ \mu(x_n,y_1) & \Lambda & \mu(x_n,y_n)\end{vmatrix}*\begin{vmatrix}v(y_1,z_1) & \Lambda & v(y_1,z_n)\\ \mathrm{M} & \mathrm{O} & \mathrm{M}\\ v(y_n,z_1) & \Lambda & v(y_n,z_n)\end{vmatrix}d\sigma(y_1)\mathrm{K}\ d\sigma(y_n)$$

$$\overline{F}(t_1-\mu_1)=\int_0^{t_1-\mu_1}\overline{F}_1(t-\mu-x)d\overline{F}_2(x)=\int\overline{F}_1(t-s)f_2(s-\mu)ds$$

$$\Theta\ \mu+x=s\Rightarrow x=s-\mu$$

$$=\iint_{s_1<s_2}\begin{vmatrix}\overline{F}_1(t_1-s_1) & \overline{F}_1(t_1-s_2)\\ \overline{F}_1(t_2-s_1) & \overline{F}_1(t_2-s_2)\end{vmatrix}*\begin{vmatrix}f_2(s_1-\mu_1) & f_2(s_1-\mu_2)\\ f_2(s_2-\mu_1) & f_2(s_2-\mu_2)\end{vmatrix}ds_2ds_1$$

Integrating the inner integral by parts, we have

$$D=\iint_{s_1<s_2}\left[\begin{vmatrix}\overline{F}_1(t_1-s_1) & f_1(t_1-s_2)\\ \overline{F}_1(t_2-s_1) & f_1(t_2-s_2)\end{vmatrix}\begin{vmatrix}f_2(s_1-\mu_1) & f_2(s_1-\mu_2)\\ \overline{F}_2(s_2-\mu_1) & \overline{F}_2(s_2-\mu_2)\end{vmatrix}\right]ds_2ds_1$$

Now the sign of first determinant is

$$\frac{\overline{F}_1(t_1-s_1)f_1(t_2-s_2)}{\overline{F}_1(t_1-s_1)\overline{F}_1(t_2-s_1)}-\frac{\overline{F}_1(t_2-s_1)f_1(t_1-s_2)}{\overline{F}_1(t_2-s_1)\overline{F}_1(t_1-s_1)}$$

Which will be same as,

$$\frac{f_1(t_2-s_2)}{\overline{F}_1(t_2-s_1)}-\frac{f_1(t_1-s_2)}{\overline{F}_1(t_1-s_1)}$$ OR same as,

$$\frac{f_1(t_2-s_2)\overline{F}_1(t_2-s_2)}{\overline{F}_1(t_2-s_2)\overline{F}_1(t_2-s_1)} - \frac{f_1(t_1-s_2)\overline{F}_1(t_1-s_2)}{\overline{F}_1(t_1-s_2)\overline{F}_1(t_1-s_1)}$$

But $\frac{f_1(t_2-s_2)}{\overline{F}_1(t_2-s_2)} \geq \frac{f_1(t_1-s_2)}{\overline{F}_1(t_1-s_2)}$ Θ if F is IFR then ratio $\frac{f(t)}{\overline{F}(t)}$ will be increasing

Also $\frac{\overline{F}_1(t_2-s_2)}{\overline{F}_1(t_2-s_1)} \geq \frac{\overline{F}_1(t_1-s_2)}{\overline{F}_1(t_1-s_1)}$ Θ if F is IFR then $\frac{\overline{F}(t+\Delta)}{\overline{F}(t)}$ will be decreasing

Thus the first determinant is non negative. A similar argument also holds second

Which implies F is IFR.

Theorem : Let F be the convolution of distribution F_1 and F_2 given by,

$$F(t) = \int_0^\infty F_1(t-x)dF_2(x)$$

Then if F_1 and F_2 are NBU, then F is NBU.

Proof :

a) we write

$$\overline{F}(x+y) = \int_0^\infty \overline{F}_2(x+y-z)dF_1(z)$$
$$= \int_0^x \overline{F}_2(x+y-z)dF_1(z) + \int_x^\infty \overline{F}_2(x+y-z)dF_1(z)$$
$$= \int_0^x \overline{F}_2(x+y-z)dF_1(z) + \int_0^\infty \overline{F}_2(y-t)dF_1(x+t)$$

But F_1 & F_2 NBU, see that $\overline{F}(x+y) \leq \overline{F}(x)\overline{F}(y)$.

Then we have $\overline{F}_2(x+y-z) \leq \overline{F}_2(y)\overline{F}_2(x-z)$.

$$\int_0^x \overline{F}_2(x+y-z)dF_1(z) \le \overline{F}_2(y)\int_0^x \overline{F}_2(x-z)dF_1(z)$$

$$= \overline{F}_2(y)\left[\overline{F}_2(x-z)F_1(z)\right]_0^x - \int_0^x F_1(z)F_2(x-z)dz$$

$$= \overline{F}_2(y)\left[F_1(x) - F(x)\right]$$

$$= \overline{F}_2(y)\left[\overline{F}(x) - \overline{F}_1(x)\right]$$

Now,

$$[\int_0^\infty \overline{F}_2(y-z)dz] * F_1(x+z)$$

$$= \overline{F}_2(y-z)F_1(x+z)\Big|_0^\infty - \int_0^\infty F_1(x+z)f_2(y-z)dz$$

$$= 1 - \overline{F}_2(y-z)F_1(x) - \int_0^\infty F_1(x+z)f_2(y-z)dz$$

$$= 1 - \overline{F}_2(y)F\left[1 - \overline{F}_1(x)\right] - \int_0^\infty \left[1 - \overline{F}_1(x+z)\right]f_2(y-z)dz$$

$$= \overline{F}_2(y)\overline{F}_1(x) + F_2(y) - \int_0^\infty f_2(y-z)dz + \int_0^\infty \overline{F}_1(x+z)f_2(y-z)dz$$

$$= \overline{F}_2(y)\overline{F}_1(x) + F_2(y) - F_2(y) + \int_0^\infty \overline{F}_1(x+z)f_2(y-z)dz$$

$$= \overline{F}_2(y)\overline{F}_1(x) + \overline{F}_1(x)[\overline{F}(y) - \overline{F}_2(y)]$$

{ Since F_1 is NBU $\Rightarrow \overline{F}_1(x+y) \le \overline{F}_1(x)\overline{F}_1(y)$ }

Thus,

$$\overline{F}(x+y) \le \overline{F}_2(y)\overline{F}(x) - \overline{F}_2(y)\overline{F}_1(x) + \overline{F}_2(y)\overline{F}_1(x) + \overline{F}_1(x)\overline{F}(y) - \overline{F}_1(x)\overline{F}_2(y)$$

$$= \overline{F}(x)\overline{F}(y) - \overline{F}(x)\overline{F}(y) + \overline{F}_2(x)\overline{F}(x) + \overline{F}_1(x)\left[\overline{F}(y) - \overline{F}_2(y)\right]$$

$$= \overline{F}(x)\overline{F}(y) - \overline{F}(x)\left[\overline{F}(y) - \overline{F}_2(y)\right] + \overline{F}_1(x)\left[\overline{F}(y) - \overline{F}_2(y)\right]$$

$$= \overline{F}(x)\overline{F}(y) - \left[\overline{F}(x) - \overline{F}_1(x)\right]\left[\overline{F}(y) - \overline{F}_2(y)\right]$$

$$\le \overline{F}(x)\overline{F}(y)$$

Repairable System :

A repairable system is a system which, after it has failed to perform properly, can be restored to satisfactory performance by any method except replacement of the entire system. For example, air conditioning equipment in an aircraft, a system of generators in a main vessel.

Frame work and Notation :

Consider a repairable system, Let $T_1, T_2, T_3, \ldots$ be the times of failure of the system and let $X_i (i = 1,2,3,\ldots)$ be the time between the failure $i-1$ and failure *i*, where T_0 is taken to be zero. The T_i and X_i are random variables. Let t_i and x_i be the corresponding observed values and let $N(t)$ be the number of failures in the time interval $(0,t]$.

The behaviour of X_i is of particular interest in reliability. Ascher and Feingold (1984) speak of happy and sad systems in which the time between failures are tending to increase and decrease, respectively. Thus the detection and estimation of trends in the X_i is a major concern. For example, it may be vital to weed out sad system which might disrupt a production process.

ROCOF (The rate of occurrence of failures) :

The ROCOF is defined by

$$\nu(t) = \frac{d}{dt} E\{N(t)\}$$

A happy system will have decreasing ROCOF and a sad system will have increasing ROCOF. However note that just because a system is happy(sad) does not necessarily mean that, in practical terms, it is satisfactory (unsatisfactory). For example – A system with very low ROCOF may be perfectly satisfactory for its planned useful life even though its ROCOF is increasing.

Note that the ROCOF, which sometimes is called the failure rate, should not be confused with the hazard function, which is also sometime known as failure rate. It is possible for the each of the X_i for a system to have non-decreasing hazard and for the system to have a decreasing ROCOF.

A natural estimator of $\nu(t)$ is $\hat{\nu}(t)$, given by

$$\hat{v}(t) = \frac{\text{no. of failures in } (t, t+\delta t)}{\delta t}$$

For some suitable δt, the choice of δt is arbitrary. But as with choosing the interval width for a histogram, the idea is to highlight the main features of the data by smoothing out the 'noise'.

Non-Homogeneous Poisson Process Model :

A stochastic point process may be regarded as a sequence of highly localized events distributed in a continuum according to some probabilistic mechanism. If we take the continuum to be time and the events as failure, it is clear that this is the natural framework for repairable systems.

Whilst it is possible to postulate a variety of point process models for the failure of a repairable system, we shall concentrate on the non homogeneous poission process (NHPP). This model is conceptually simple, it can model happy and sad systems and the relevant statistical methodology is well developed and easy to apply

The assumptions for a NHPP are as for the poisson process except that the ROCOF varies with time rather than being a constant. The assumptions of Poisson process are

1. Failures occurring in disjoint time intervals are statistically independent.
2. The failure rate (number of failures per unit time) is constant and so does not depend upon the particular time interval examined.

Consider a NHPP with time dependent ROCOF $v(t)$ (this is sometime called intensity function or Pril rate), then the numbers of failures in the time interval $(t_1, t_2]$ has a Poisson distribution

with mean $\lambda = \frac{1}{x!} e^{-\int_{t_1}^{t_2} v(t)dt} \left[\int_{t_1}^{t_2} v(t)dt \right]^x$

Thus the probability of number of failure in (t_1, t_2) is $\exp\left\{ -\int_{t_1}^{t_2} v(t)dt \right\}$.

By choosing a suitable parametric form for $v(t)$, one obtain a flexible model for the failures of a repairable system in a 'minimal repair' setup; that is, when only a small proportion of the constituent parts of the system are replaced or repair.

Suppose we observe a system for the time interval $[0,t_0]$ with failures occurring at $t_1, t_2, t_3, \ldots t_n$. The likelihood may be obtained as follows:

The probability of observing no failures in $(0,t_1)$, one failure in $(t_1, t_1+\delta t_1)$, no failures in $(t_1+\delta t_1, t_2)$, one failure in $(t_2, t_2+\delta t_2)$ and so on up to no failures in $(t_n+\delta t_n, t_n)$, (for small $\delta t_1, \delta t_2, \ldots \delta t_n$) is :

$$\left\{\exp\left(-\int_0^{t_1} \nu(t)dt\right)\right\}\nu(t_1)\delta t_1\left\{\exp\left(-\int_{t_1+\delta t_1}^{t_2} \nu(t)dt\right)\right\}\nu(t_2)\delta t_2 \ldots \left\{\exp\left(-\int_{t_n+\delta t_n}^{t_0} \nu(t)dt\right)\right\}$$

Dividing throughout by $\delta t_1, \delta t_2, \ldots \delta t_n$ and letting $\delta t_i \to 0$ $(i=1,2\ldots n)$ gives the likelihood function

$$L=\left\{\prod_{\nu=1}^{n} \nu(t_i)\right\}\exp\left(-\int_0^{t_0} \nu(t)dt\right) \qquad (1)$$

and the log likelihood is thus,

$$\log L=\sum_{i=1}^{n} \log \nu(t_i)-\int_0^{t_0} \nu(t)dt \qquad (2)$$

Another possible scheme for observation of a repairable is to observe the system until the *n*th failure. Expression (1) and (2) still apply but with t_0 replaced by t_n. Sometimes, the failure times are not observed and only the number of failures within non-overlapping time intervals are available. Suppose, for example, that $n_1, n_2, n_3, \ldots n_m$ failures have been observed in non-overlapping time intervals $(a_1, b_1], (a_2, b_2] \ldots, (a_m, b_m]$ then the likelihood is,

$$L=\left\{\exp\left(-\int_{a_1}^{b_1} \nu(t)dt\right)\frac{\left(\int_{a_1}^{b_1} \nu(t)dt\right)^{n_1}}{n_1!}\right\}\ldots\left\{\exp\left(-\int_{a_m}^{b_m} \nu(t)dt\right)\frac{\left(\int_{a_m}^{b_m} \nu(t)dt\right)^{n_m}}{n_m!}\right\}$$

$$=\exp\left\{-\sum_{i=1}^{m}\int_{a_i}^{b_i} \nu(t)dt\right\}\prod_{i=1}^{m}\frac{\left(\int_{a_i}^{b_i} \nu(t)dt\right)^{n_i}}{n_i!}$$

Thus the log-likelihood (apart from additive constant) is

$$\log L = \sum_{i=1}^{m} \left\{ n_i \log \int_{a_i}^{b_i} \nu(t)dt - \int_{a_i}^{b_i} \nu(t)dt \right\} \tag{3}$$

Therefore, once $\nu(t)$ has been specified, it is straight forward to use the likelihood based method to obtain ML estimates for any unknown parameters inherent in the specification of $\nu(t)$. We shall concentrate on two simple choices of $\nu(t)$ that give monotonic ROCOF:

1) **<u>Log-Linear ROCOF</u>**

$$\nu_1(t) = \exp(\beta_0 + \beta_1 t)$$

2) **<u>Weibull process</u>**

$$\nu_2(t) = \gamma\delta t^{\delta-1}; \quad \gamma > 0, \delta > 0$$

<u>NHPP with Log-Linear ROCOF</u> :

$$\nu_1(t) = \exp(\beta_0 + \beta_1 t) \tag{4}$$

it gives a simple model to describe a happy system $(\beta_1 < 0)$ or a sad system $(\beta_1 > 0)$. Also if β_1 is near zero $\nu_1(t)$ approximate a linear trend in ROCOF over short time period.

Here we discuss some likelihood based statistical methods for fitting a NHPP with $\nu_1(t)$ to a set of repairable system data.From (2), we have

$$\log L = \sum_{i=1}^{n} \log \nu(t_i) - \int_{0}^{t_0} \nu(t)dt$$

Substituting the value of $\nu(t)$, we have

$$\log L = \sum_{i=1}^{n} (\beta_0 + \beta_1 t_i) - \int_{0}^{t_0} e^{\beta_0 + \beta_1 t} dt$$
$$= n\beta_0 + \beta_1 \sum_{i=1}^{n} t_i - e^{\beta_0} \int_{0}^{t_0} e^{\beta_1 t} dt \quad .$$
$$= n\beta_0 + \beta_1 \sum_{i=1}^{n} t_i - e^{\beta_0} \frac{\left(e^{\beta_1 t_0} - 1\right)}{\beta_1}$$

To obtain ML estimate of β_0 and β_1, we have

$$\frac{\partial \log L}{\partial \beta_0} = n - \frac{1}{\beta_1} e^{\beta_0}\left(e^{\beta_1 t_0} - 1\right) \qquad (1)$$

$$\frac{\partial \log L}{\partial \beta_0} = \sum t_i - e^{\beta_0}\left[\frac{\beta_1 e^{\beta_1 t_0} t_0 - \left(e^{\beta_1 t_0} - 1\right) * 1}{\beta_1^2}\right] = 0$$

$$e^{\beta_0} = \frac{n\beta_1}{\left(e^{\beta_1 t_0} - 1\right)} \qquad (2)$$

from (1), putting the value of e^{β_0} in (2), we get

$$\sum t_i - \frac{n\beta_1}{\left(e^{\beta_1 t_0} - 1\right)}\left[\frac{t_0 \beta_1 e^{\beta_1 t_0} - \left(e^{\beta_1 t_0} - 1\right)}{\beta_1^2}\right] = 0$$

i.e.

$$\sum t_i - n t_0 \frac{e^{\beta_1 t_0}}{\left(e^{\beta_1 t_0} - 1\right)} + n\beta_1^{-1} = 0$$

i.e.

$$\sum t_i - n t_0 \left\{1 - e^{-\beta_1 t_0}\right\}^{-1} + n\beta_1^{-1} = 0$$

Solving this equation we can get the value of β_1 and by putting the obtained value of β_1 we can get lo

NHPP with ROCOF ν_2:

If $\nu(t)$ is of the form

$$\nu(t) = \gamma \delta t^{\delta - 1} \qquad (*)$$

then putting this in equation(2)

i.e. $$\log L = \sum_{i=1}^{n} \log \nu(t_i) - \int_0^{t_0} \nu(t)\, dt \qquad (**)$$

from (*) & (**)

$$\log L = \sum_{i=1}^{n} \log\left(\gamma\delta t_i^{\,\delta-1}\right) - \int_0^{t_0} \gamma\delta t^{\,\delta-1} dt$$

$$= n\log\gamma + n\log\delta + (\delta-1)\sum_{i=1}^{n}\log t_i - \frac{\gamma\delta t_0^{\,\delta-1}}{\delta}$$

$$= n\log\gamma + n\log\delta + (\delta-1)\sum\log t_i - \frac{\gamma\delta t_0^{\,\delta-1}}{\delta}$$

$$= n\log\gamma + n\log\delta + (\delta-1)\sum\log t_i - \gamma t_0^{\,\delta}$$

$$\frac{\partial \log L}{\partial\gamma} = \frac{n}{\gamma} - t_0^{\delta} = 0 \Rightarrow \hat{\gamma} = \frac{n}{t_0^{\delta}}$$

Now

$$\frac{\partial \log L}{\partial\delta} = \frac{n}{\delta} + \sum\log t_i - \gamma t_0^{\delta}\log t_0 = 0$$

$$or$$

$$\frac{n}{\delta} + \sum\log t_i - n\log t_0 = 0 \qquad \left[\hat{\gamma} = \frac{n}{t_0^{\delta}}\right.$$

$$\Rightarrow \hat{\delta} = \frac{n}{n\log t_0 - \sum_{i=1}^{n}\log t_i}$$

Accelerated Life Tests :

Suppose that under the constant application of a single stress V_i a device has an exponential failure distribution given by,

$$f(t,\lambda_i)=\lambda_i e^{-\lambda_i t}$$

Where λ_i is constant hazard rate under the stress V_i. If $\theta_i = 1/\lambda_i$, then θ_i is the mean time to failure under the stress V_i.

We may have following relationship between λ_i and V_i.

1. **The power rule model** :

   It is a device via consideration of kinetic theory and activation energy and may be applied to paper-impregnated dielectric capacitors.

   In $V_i's$ model, we have

$$\frac{1}{\lambda_i}=\frac{c}{V_i^p};c>0$$

$$\Rightarrow \lambda_i=\frac{1}{c}V_i^p$$

   Where V_i is voltage and p and c are estimated.

2. **The Reaction Rate Model** : **(closed for semiconductor)**

   Here ,

$$\lambda_i=\exp\left(A-\frac{B}{V_i}\right)$$

   Where V_i is temperature stress. A & B are to be estimated.

3. **The Eyring Model** :

   Here,

$$\lambda_i=V_i\exp\left(A-\frac{B}{V_i}\right)$$

   Where V_i is temperature.

Conducting Accelerated Life Test :

Suppose that k values of stress $V_j, j = 1,2 \Lambda, k$, are chosen randomly that are to be applied on a device which has under a stress V_j has an exponential failure distribution with scale parameter $\lambda_j = 1/\theta_j$. Further suppose that while applying each stress V_i let n_i devices are put to test and the test is terminated after failure of r_i items, thus the data $(V_i, n_i, r_i, \hat{\theta}_i)$, where $\hat{\theta}_i$ is an estimator of θ_i .

It is known that MLE as well as UMVUE of θ_i , $\hat{\theta}_i$ is given as

$$\hat{\theta}_i = \frac{1}{r_i} \sum_{j=1}^{r_i} t_{ji} + (n_i - r_i) tr_i i \tag{1}$$

The *pdf* of $\hat{\theta}_i$ is a gamma density with shape parameter r_i , so that

$$g(\hat{\theta}_i) = \frac{1}{\overline{)r_i}} \left(\frac{r_i}{\theta_i} \right) \left(\hat{\theta}_i \right)^{r-1} \exp\left(\frac{-r_i \hat{\theta}_i}{\theta_i} \right) \tag{2}$$

Estimation under power rule model :

Here $$\theta_i = \frac{c}{\left(V_i \big/ \overline{V} \right)^p}$$

where $\overline{V} = \prod_{i=1}^{k} V_i^{\,r_i \big/ \sum_{i=1}^{k} r_i}$ Is weighted geometric mean of V_i 's.

The likelihood function for C & P is

$$L(C, P / \hat{\theta}) = \prod_{i=1}^{k} \frac{1}{\overline{)r_i}} \left[\frac{r_i}{C} \left(\frac{V_i}{\overline{\overline{V}}} \right)^P \right]^{r_i} \left(\hat{\theta}_c \right) \exp\left[-\frac{r_i \hat{\theta}_i}{C} \left(\frac{V_i}{\overline{\overline{V}}} \right)^P \right]$$

where $\hat{\theta} = \left[\hat{\theta}_1, \hat{\theta}_2, \text{K}, \hat{\theta}_k \right]$

$$\ln L = \sum_{i=1}^{k}\left[\left(-\log\overline{)r_i}\right) + r_i\left[\log r_i - \ln C + P\log\left(\frac{V_i}{\overline{\overline{V}}}\right)\right]\right] + \left(r_i - 1\right)\ln\hat{\theta}_i - \frac{r_i\hat{\theta}_i}{C}\left(\frac{V_i}{\overline{\overline{V}}}\right)^P$$

$$= \sum\left[-\log\overline{)r_i} + r_i\ln r_i - r_i\ln C + r_i P\log\left(\frac{V_i}{\overline{\overline{V}}}\right) + \left(r_i - 1\right)\log\hat{\theta}_i - \frac{r_i\hat{\theta}_i}{C}\left(\frac{V_i}{\overline{\overline{V}}}\right)^P\right]$$

$$\frac{\partial \ln L}{\partial C} = -\frac{\sum r_i}{C} + \frac{1}{C^2}\sum r_i\hat{\theta}_i\left(\frac{V_i}{\overline{\overline{V}}}\right)^P = 0$$

$$\hat{C} = \frac{\sum_{i=1}^{k} r_i\hat{\theta}_i\left(\frac{V_i}{\overline{\overline{V}}}\right)^P}{\sum_{i=1}^{k} r_i}$$

Further

$$\frac{\partial \ln L}{\partial P} = \sum_{i=1}^{k} r_i\left[\log\left(\frac{V_i}{\overline{\overline{V}}}\right)\right] - \sum_{i=1}^{k}\frac{r_i\hat{\theta}_i}{C}\left(\frac{V_i}{\overline{\overline{V}}}\right)^P\log\left(\frac{V_i}{\overline{\overline{V}}}\right)$$

$$= C\sum_{i=1}^{k} r_i\log\left(\frac{V_i}{\overline{\overline{V}}}\right) - \sum_{i=1}^{k} r_i\hat{\theta}_i\left(\frac{V_i}{\overline{\overline{V}}}\right)^P\log\left(\frac{V_i}{\overline{\overline{V}}}\right) = 0$$

Substituting the value of $\hat{C}$, we get

$$\left(\frac{V_i}{\overline{\overline{V}}}\right)^P\frac{\partial \ln L}{\partial P} = \sum_{i=1}^{k} r_i\log\left(\frac{V_i}{\overline{\overline{V}}}\right)\left\{C - \hat{\theta}_i\left(\frac{V_i}{\overline{\overline{V}}}\right)^P\right\}$$

$$= \sum_{i=1}^{k} r_i\log\left(\frac{V_i}{\overline{\overline{V}}}\right)\left\{\frac{\sum r_i\hat{v}_i\left(\frac{V_i}{\overline{\overline{V}}}\right)^P}{\sum r_i} - \hat{\theta}\left(\frac{V_i}{\overline{\overline{V}}}\right)^P\right\} = 0$$

The above equations can be solved simultaneously by any appropriate numerical method.